Q.No.→ State and Prove Cauchy's Inequality.

Ans. Statement:- If $f(z)$ is analytic within $C$, given by $|z-z_0| = \rho$ and if $|f(z)| \leq M$ on $C$, then,

$$|f^n(z_0)| \leq \frac{M\lfloor n}{\rho^n}$$

Proof:- We know that,

$$f^{n}(z_0) = \frac{\underline{|n}}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

$$\therefore |f^{n}(z_0)| = \left|\frac{\underline{|n}}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}\right|$$

$$\leq \frac{\underline{|n}}{|2\pi i|}\int_C \frac{|f(z)|\,|dz|}{|z-z_0|^{n+1}}$$

$$\leq \frac{\underline{|n}}{2\pi}\frac{M}{\rho^{n+1}}\int_0^{2\pi} R\,d\theta$$ Since, $z = Re^{i\theta}$

$|dz| = |iRe^{i\theta}\,d\theta| = R\,d\theta$

$$\leq \frac{\underline{|n}}{\cancel{2\pi}}\frac{M}{\rho^{n+1}}\cancel{2\pi}\rho$$

$$\leq \frac{M\underline{|n}}{\rho^{n}}$$

$$\therefore |f^{n}(z_0)| \leq \frac{M\underline{|n}}{\rho^{n}}.$$

M.Sc

QNo.→ State and Prove Morera's theorem.

Ans.→ Statement:- Let $f(z)$ be Continuous in a Simply Connected domain D and let

$$\int_C f(z)dz = 0.$$

Where C any rectifiable closed Jordan Curve in D. Then $f(z)$ is analytic in D.

Proof:- Let $z_0$ be a fixed Point and z any variable Point in D. and let $C_1, C_2$ be any two Continuous rectifiable arcs in D Joining $z_0$ to z. Let C denote the closed rectifiable Continuous curve Consisting of $C_1$ & $-C_2$.

$$\therefore \int_C f(z)dz = \int_{C_1 - C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{-C_2} f(z)dz$$

$$= \int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0$$

$$\therefore \int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

Hence, the integral along any curve in D Joining $z_0$ to z is the same.

$$\text{Let } F(z) = \int_{z_0}^{z} f(\xi_1)\,d\xi_1 \quad \text{— ①}$$

The above eqn. is Justified since the integral is Path-independent.

Let $z+h$ be any Point near the Point z.

$$\therefore F(z+h) = \int_{z_0}^{z+h} f(\xi)\, d\xi \quad \text{—— (2)}$$

Subtracting (1) from (2), we get

$$F(z+h) - F(z) = \int_{z_0}^{z+h} f(\xi)\, d\xi - \int_{z_0}^{z} f(\xi)\, d\xi$$

$$= \int_{z_0}^{z+h} f(\xi)\, d\xi + \int_{z}^{z_0} f(\xi)\, d\xi$$

$$= \int_{z}^{z+h} f(\xi)\, d\xi \quad \text{—— (3)}$$

Since, the integrals in (3) are Path-independent, we may take the integral along a straight line $z$ to $z+h$.

$$\therefore \frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h}\int_{z}^{z+h} f(z)\, d\xi - \frac{hf(z)}{h}.$$

$$= \frac{1}{h}\left[\int_{z}^{z+h} f(\xi)\, d\xi - f(z)\int_{z}^{z+h} dz\right]$$

$$= \frac{1}{h}\int_{z}^{z+h} [f(\xi) - f(z)]\, d\xi.$$

Since, $f(\xi)$ is Continuous at $z$, for a given $\epsilon > 0$, there exist $\delta > 0$ such that

$$|f(\xi) - f(z)| < \epsilon \quad \forall\ \xi \text{ Satisfying}$$

$$|\xi - z| < \delta \quad \text{—— (6) (4)}$$

We now Choose $h$ such that

$|h| < \delta$.

$\therefore$ (4) is satisfied for every $\xi$ line on the line segment joining $z$ to $z+h$,

$$\therefore \left| \frac{F(z+h)-F(z)}{h} - f(z) \right| = \left| \frac{1}{h} \int_{z}^{z+h} \{f(\xi) - f(z)\} d\xi \right|$$

$$\leq \frac{1}{|h|} \int_{z}^{z+h} |f(\xi) - f(z)| \, |d\xi|$$

$$< \frac{\epsilon}{|h|} \cdot |z+h-z| = \frac{\epsilon}{|h|} \cdot |h| = \epsilon$$

$$\therefore \lim_{h \to 0} \frac{F(z+h)-F(z)}{h} = f(z)$$

$$\therefore F'(z) = f(z).$$

Hence, $F(z)$ ~~Possesses~~ passes the derivative $f(z)$ at every $z \in D$.

$\therefore$ $F(z)$ is analytic in $D$ and its derivative of an analytic function is analytic.

Hence, $f(z)$ is analytic in $D$.